

# Strings in the Quantized pp-wave Backgrounds from Membrane

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## Abstract

In this paper we study strings with quantized masses in the pp-wave backgrounds. We obtain these strings from the membrane theory. For achieving to this, one of the membrane and one of the spacetime directions will be identified and wrapped. From the action of strings in the quantized pp-wave backgrounds, we obtain its mass dual action. Some properties of the closed and open strings in these backgrounds will be studied.

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# 1 Introduction

It has been known that strings in the pp-wave NS backgrounds are exactly solvable [1]. The same is true for pp-waves on the *Ramond-Ramond* backgrounds. Solvability in this context means that it is possible to find explicitly the solutions of the classical string equations, perform a canonical quantization and determine the Hamiltonian operator. It was recently pointed out that the plane wave metric supported by an *RR* 5-form background [2]

$$\begin{aligned} ds^2 &= 2dX^+dX^- - \mu^2 \sum_{I=1}^8 X^I X^I (dX^+)^2 + \sum_{I=1}^8 dX^I dX^I, \\ F_{+1234} &= F_{+5678} = 2\mu, \end{aligned} \tag{1}$$

provides examples of exactly solvable string models [3]. That is, the string action becomes quadratic in the light-cone gauge  $X^+ = x^+ + p^+ \tau$  [1, 3, 4].

Some properties of strings propagating in such plane-wave backgrounds have been previously investigated, in particular in the Refs.[3, 5]. This background is related, by a special limit [6], to the  $AdS_5 \times S^5$  background [7]. The solvability of string theory in this background has some common features with string theory on  $AdS_5 \times S^5$  [8]. Another important application of string theory in this background is that, it tests AdS/CFT correspondence beyond the supergravity approximation [9].

In the other side we have the membrane theory. It is characterized by the 11-dimensional spacetime [10]. For more review also see the Refs.[11, 12]. In fact, the membrane theory, by compactification and double dimensional reduction, provides an enlarged framework for the study of strings [13].

Let one of the spatial directions of the membrane and of the spacetime be identified and compactified on a circle. In this case, the membrane can be viewed as infinite number of strings in the pp-wave backgrounds with both positive and negative quantized masses. This implies that the canonical quantization of the membrane theory imposes the canonical quantization on the strings in the pp-wave backgrounds. Therefore, the radius of the compactification will be fixed.

The action of these strings (i.e.,  $S$ ) describes interactions between the worldsheets fields with opposite masses. By a transformation on the action  $S$ , we obtain another useful action (i.e.,  $S'$ ), which contains interactions between the worldsheets fields with the same masses. Under some transformations on the worldsheets fields, these actions are invariant or transform to each other.

We shall obtain the algebras and the Hamiltonians of the closed and open strings. These Hamiltonians are normal ordered with respect to one of the two indices that appear in each

oscillator.

Note that we shall consider only the bosonic membrane. Therefore, we obtain only bosonic strings. Generalization to the supersymmetric case is straightforward. It is sufficient to consider supermembrane instead of the bosonic membrane. Since our formulation comes from the membrane theory, we shall consider the eleven dimensional spacetime.

This paper is organized as the follows. In section 2, we obtain the action of strings in the pp-wave backgrounds, its quantization and its corresponding Hamiltonian from the compactified membrane theory. In section 3, we study the mass-dual theory and the symmetries of the actions  $S$  and  $S'$ . In section 4, we study the massive closed strings. In section 5, the massive open strings will be analyzed.

## 2 Strings from membrane

It is possible to choose a gauge that identifies the eleventh dimension of the spacetime with the third dimension of the membrane worldvolume. This gauge is  $Z^{10} = \rho$ , where  $\rho, \sigma$  and  $\tau$  are coordinates of the membrane worldvolume [12]. This gauge also removes the cosmological constant from the membrane action. By the procedure of the Ref.[12], the light-cone coordinates  $Z^\pm = \frac{1}{\sqrt{2}}(Z^0 \pm Z^9)$  can also put away from the action. Therefore, the membrane action becomes

$$S = -\frac{T_2}{2} \int d^2\sigma d\rho \eta^{AB} \partial_A Z^I \partial_B Z^I, \quad (2)$$

where  $I \in \{1, 2, \dots, 8\}$ . The metric of the membrane worldvolume is  $\eta_{AB} = \text{diag}(-1, 1, 1)$  and  $T_2$  is the membrane tension. The equation of motion of the membrane extracted from this action is

$$(-\partial_\tau^2 + \partial_\sigma^2 + \partial_\rho^2) Z^I(\tau, \sigma, \rho) = 0. \quad (3)$$

If the  $\rho$ -direction of the membrane is wrapped around a circle with the radius  $R$ , then the  $Z^{10}$ -direction also is compact on the same circle. In fact, this is double dimensional compactification, which compactifies both the spacetime and the worldvolume on the same circle. Upon dimensional reduction by shrinking the circle, the  $Z^{10}$ -direction of the spacetime and the wrapped spatial dimension of the membrane will be shrunk, for producing strings in ten dimension.

Now consider the set of functions

$$\left\{ \exp\left(\frac{in\rho}{R}\right) \right\}, n \in Z, \quad (4)$$

where the range of  $\rho$  is  $0 \leq \rho \leq 2\pi R$ . This set of functions forms a complete set. Therefore, we can expand  $Z^I(\tau, \sigma, \rho)$ , in terms of them, i.e.,

$$Z^I(\tau, \sigma, \rho) = \sum_{n=-\infty}^{\infty} X_n^I(\tau, \sigma) q_n(\rho), \quad (5)$$

where  $q_n(\rho) = a_n \exp(\frac{in\rho}{R})$ . Therefore, the membrane may be viewed as a “tower of strings” with the coordinates  $\{X_n^I(\tau, \sigma) | n \in \mathbb{Z}\}$ . As expected by compactification, the coordinate  $Z^I(\tau, \sigma, \rho)$  with respect to  $\rho$  is periodic, i.e.,  $Z^I(\tau, \sigma, \rho + 2\pi R) = Z^I(\tau, \sigma, \rho)$ . Since the membrane coordinate  $Z^I(\tau, \sigma, \rho)$  is real, we should have

$$\begin{aligned} a_n^\dagger &= a_{-n}, \\ X_n^{I\dagger}(\tau, \sigma) &= X_{-n}^I(\tau, \sigma). \end{aligned} \quad (6)$$

The membrane action (2) gives the momentum conjugate to the coordinate  $Z^I(\tau, \sigma, \rho)$  as

$$\Pi^I(\tau, \sigma, \rho) = T_2 \sum_{n \in \mathbb{Z}} q_n(\rho) \partial_\tau X_n^I(\tau, \sigma). \quad (7)$$

Quantization of the degrees of freedom is achieved by imposing the equal time canonical commutation relation

$$\left[ Z^I(\tau, \sigma, \rho), \Pi^J(\tau, \sigma', \rho') \right] = i\delta^{IJ} \delta(\sigma - \sigma') \delta(\rho - \rho'). \quad (8)$$

In other words, it is

$$T_2 \sum_{n \in \mathbb{Z}} \sum_{n' \in \mathbb{Z}} q_n(\rho) q_{n'}(\rho') \left[ X_n^I(\tau, \sigma), \partial_\tau X_{n'}^J(\tau, \sigma') \right] = i\delta^{IJ} \delta(\sigma - \sigma') \delta(\rho - \rho'). \quad (9)$$

The delta functions in the right hand side imply that,

$$\left[ X_n^I(\tau, \sigma), \partial_\tau X_{n'}^J(\tau, \sigma') \right] = 2\pi\alpha' i\delta^{IJ} \delta(\sigma - \sigma') \delta_{n+n', 0}. \quad (10)$$

The factor  $2\pi\alpha'$  is introduced for later purposes. We shall see that, this equation gives the quantization of the strings coordinates  $\{X_n^I(\tau, \sigma)\}$ . Therefore, the equation (9) reduces to

$$\sum_{n \in \mathbb{Z}} q_n(\rho) q_{-n}(\rho') = \frac{1}{2\pi\alpha' T_2} \delta(\rho - \rho'). \quad (11)$$

Now consider the orthogonality equation

$$\int_0^{2\pi R} d\rho q_n(\rho) q_{n'}(\rho) = 2\pi R a_n a_{-n} \delta_{n+n', 0}. \quad (12)$$

According to this integral, the completeness of the functions  $\{q_n(\rho)\}$  gives

$$\frac{1}{2\pi R} \sum_{n \in \mathbb{Z}} \frac{1}{a_n a_{-n}} q_n(\rho) q_{-n}(\rho') = \delta(\rho - \rho'). \quad (13)$$

Without lose of generality, we assume that  $a_n a_{-n} = 1$  for all values of the integer  $n$ . In this case, the equations (11) and (13) give the relation

$$R = \frac{1}{4\pi^2 \alpha' T_2}. \quad (14)$$

Therefore, the radius of compactification is fixed. That is, for the fixed parameters  $\alpha'$  and  $T_2$ , we can not decompactify or shrink the  $\rho$ -direction, to a circle with larger or smaller radius than  $R$ .

Finally, the action becomes

$$S = -\frac{1}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \int d^2\sigma \left( \eta^{ab} \partial_a X_n^I \partial_b X_{-n}^I + \mu_n^2 X_n^I X_{-n}^I \right), \quad (15)$$

where the worldsheets metric is  $\eta_{ab} = \text{diag}(-1, 1)$  and

$$\mu_n = \frac{n}{R} = n(4\pi^2 \alpha' T_2). \quad (16)$$

This implies that  $\mu_n$  is the momentum conjugate to the compact coordinate  $\rho$ , and  $n$  is momentum number. The action (15) describes infinite number of strings that interact with each other. Strings with positive and negative masses, where the absolute values of these masses are equal, couple to each other. There is no coupling between the worldsheets fields with the same masses. Since the quantity  $4\pi^2 \alpha' T_2$  is quantum of the strings masses, one interpretation is that, the membrane is a collection of infinite number of strings in the quantized pp-wave backgrounds.

The equation of motion of a string with mass number  $n$ , extracted from this action is

$$(-\partial_\tau^2 + \partial_\sigma^2 - \mu_n^2) X_n^I(\tau, \sigma) = 0. \quad (17)$$

This equation also can be obtained from the equation (5) and the membrane equation of motion (3).

The integral (12) and the equation (5) give the strings coordinates in terms of the membrane coordinates

$$X_n^I(\tau, \sigma) = 2\pi\alpha' T_2 \int_0^{2\pi R} d\rho q_{-n}(\rho) Z^I(\tau, \sigma, \rho). \quad (18)$$

Therefore, by an appropriate solution of the membrane, we can obtain solutions of the massive strings. But we shall obtain  $X_n^I(\tau, \sigma)$  from the massive string equation (17).

## The Hamiltonian

The action (15) gives the momentum conjugate to the worldsheet field  $X_n^I(\tau, \sigma)$  as

$$\Pi_n^I(\tau, \sigma) = \frac{1}{2\pi\alpha'} \partial_\tau X_{-n}^I(\tau, \sigma). \quad (19)$$

This implies that, strings with opposite mass numbers have conjugation with each other. According to this conjugate momentum, the corresponding Hamiltonian to the action (15) is

$$H = \frac{1}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \int_0^{2\pi\alpha' p^+} d\sigma \left( \partial_\tau X_n^I \partial_\tau X_{-n}^I + \partial_\sigma X_n^I \partial_\sigma X_{-n}^I + \mu_n^2 X_n^I X_{-n}^I \right). \quad (20)$$

If we write  $H = \sum_{n \in \mathbb{Z}} H_n$ , the condition (6) implies that each  $H_n$  (i.e., the Hamiltonian of the string with mass number  $n$ ) is Hermitian. This Hamiltonian also can be obtained from the membrane Hamiltonian.

According to the equation (10) we have the following canonical quantization for the degrees of freedom of the action (15)

$$\left[ X_n^I(\tau, \sigma), \Pi_{n'}^J(\tau, \sigma') \right] = i\delta^{IJ}\delta(\sigma - \sigma')\delta_{nn'}. \quad (21)$$

This quantization will be used for obtaining the algebra of the string modes.

The Hamiltonian (20) and the action (15) under the transformation  $X_n^I \rightarrow X_{-n}^I$ , are invariant. In the next section we shall study a general symmetry transformation of this Hamiltonian and the action (15).

## 3 Mass duality

Under some transformations the action (15) remains invariant. Furthermore, there are some other transformations that change this action to another useful action. Consider the transformation

$$X_n^I \rightarrow Y_n^I = A_{(n)J}^I X_n^J + B_{(n)J}^I X_{-n}^J, \quad (22)$$

where  $A_n$  and  $B_n$  are  $8 \times 8$  matrices. Now change  $n \rightarrow -n$  and compare the result with the conjugation of the above transformation (i.e., perform dagger on it), we obtain

$$\begin{aligned} A_n^\dagger &= A_{-n}, \\ B_n^\dagger &= B_{-n}. \end{aligned} \quad (23)$$

These also give  $Y_n^{I\dagger} = Y_{-n}^I$ . Note that this kind of conjugation only acts on the index  $n$ , therefore, spacetime indices under it are neutral.

We now consider two cases. One case is that the action (15) and the corresponding Hamiltonian (20) remain invariant and the other case is that they transform to another useful action and Hamiltonian.

### The invariance case

For this case the matrices  $A_n$  and  $B_n$  satisfy the equations

$$\begin{aligned} A_n^T A_{-n} + B_{-n}^T B_n &= \mathbf{1}, \\ A_n^T B_{-n} + B_{-n}^T A_n &= \mathbf{0}, \end{aligned} \quad (24)$$

where  $n$  is any integer number. According to these equations, these matrices also satisfy the following equations

$$\begin{aligned} A_n A_{-n}^T + B_n B_{-n}^T &= \mathbf{1}, \\ A_n B_n^T + B_n A_n^T &= \mathbf{0}. \end{aligned} \quad (25)$$

Note that these equations are not independent of the equations (24).

Define the matrix  $\mathcal{M}_n$  as

$$\mathcal{M}_n = \begin{pmatrix} A_n & B_n \\ B_{-n} & A_{-n} \end{pmatrix}. \quad (26)$$

Therefore, the equations (24) and (25) can be written as in the following

$$\mathcal{M}_n^T \mathcal{M}_{-n} = \mathcal{M}_n \mathcal{M}_{-n}^T = \mathbf{1}. \quad (27)$$

These are invariance conditions of the action (15) and the Hamiltonian (20). For the zero mass case,  $\mathcal{M}_0$  is an orthogonal matrix.

### The mass-dual action

Let the matrix  $\mathcal{M}_n$  satisfy the equations

$$\mathcal{M}_n^T \mathcal{M}_{-n} = \mathcal{M}_n \mathcal{M}_n^T = \mathbf{J}, \quad (28)$$

where the matrix  $\mathbf{J}$  has the definition

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_{8 \times 8} \\ \mathbf{1}_{8 \times 8} & \mathbf{0} \end{pmatrix}. \quad (29)$$

In this case, the action (15) and the Hamiltonian (20) transform to the action

$$S' = -\frac{1}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \int d^2\sigma \left( \eta^{ab} \partial_a X_n^I \partial_b X_n^I + \mu_n^2 X_n^I X_n^I \right), \quad (30)$$

and the Hamiltonian

$$H' = \frac{1}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \int_0^{2\pi\alpha' p^+} d\sigma \left( \partial_\tau X_n^I \partial_\tau X_n^I + \partial_\sigma X_n^I \partial_\sigma X_n^I + \mu_n^2 X_n^I X_n^I \right). \quad (31)$$

We call  $S'$  as “*mass-dual action*” of the action  $S$ . The action  $S'$  implies that, for each mass number  $n$  we have a light-cone action, which has been produced after the usual gauge fixing [3, 14]. In other words, for each mass number “ $n$ ”, there is a string in the pp-wave background  $\mu_n$ . According to this action, only strings with the same mass couple to each other, while in the action  $S$  strings with opposite masses have coupling.

The action  $S'$  can be splitted to  $S' = S'_0 + S'_+ + S'_-$ , where  $S'_0$ ,  $S'_+$  and  $S'_-$  correspond to zero, positive and negative masses, respectively. There is also  $S'_+ = S'_-$ . Similarly, the Hamiltonian  $H'$  can be written as  $H' = H'_0 + \sum_{n=1}^{\infty} (H'_n + H'_{-n})$ . The condition (6) implies that each  $H'_n$  (except  $H'_0$ ) separately is not Hermitian. Therefore, we should consider at least a system of two strings with equal but opposite masses. This system has the Hamiltonian  $H'_n + H'_{-n}$ , which is Hermitian.

According to the action  $S'$ , the momentum conjugate to the coordinate  $X_n^I$  is

$$\Pi_n^I(\tau, \sigma) = \frac{1}{2\pi\alpha'} \partial_\tau X_n^I(\tau, \sigma). \quad (32)$$

Therefore, the Hamiltonian  $H'$  also can be extracted from the action  $S'$ , as expected. Since  $\Pi_n^I = \Pi_{-n}^I$ , the quantization of the action  $S$ , i.e., the equation (21), leads to the quantization of the action  $S'$  as in the following

$$\left[ X_n^I(\tau, \sigma), \Pi_{n'}^J(\tau, \sigma') \right] = i\delta^{IJ}\delta(\sigma - \sigma')\delta_{n+n', 0}. \quad (33)$$

This comes from the mass duality, that is, some properties of one theory can be extracted from its dual theory.

Look at the usual string theory, that is, the worldsheet fields are massless. Under the  $T$ -duality transformations we have the exchange  $X^I \leftrightarrow X'^I$ , where  $X^I = X_L^I + X_R^I$  and  $X'^I = X_L^I - X_R^I$ . Then, the string Hamiltonian is invariant. Furthermore, the conjugate momentum  $P^I = \frac{1}{2\pi\alpha'} \partial_\tau X^I$  and the momentum  $p^I = p_L^I + p_R^I$  of the string action, exchange with the conjugate momentum  $P'^I = \frac{1}{2\pi\alpha'} \partial_\tau X'^I$  and the momentum  $\frac{1}{\alpha'} L^I = p_L^I - p_R^I$  of the  $T$ -dual action of string. Similar to the coordinates  $X^I$  and  $X'^I$ , we have  $X_n^I$  and  $X_{-n}^I$ . Under the exchange  $X_n^I \leftrightarrow X_{-n}^I$ , the Hamiltonian (20) is invariant and the conjugate momenta  $\Pi_n^I$  and  $\Pi_n'^I$  change to each other. We shall see that the momenta  $p_n^I$  and  $p_n'^I = p_{-n}^I$  also change to each other. Since we have the change  $n \rightarrow -n$ , according to the equation (16) the mass  $\mu_n$  transforms to  $-\mu_n$ . For these reasons we call the action (30) as the mass dual action of the action (15).



## Transformations of the mass-dual action

Application of the transformation (22) on the action  $S'$  and on the Hamiltonian  $H'$  leads to the two interesting cases. One case is that  $S'$  and  $H'$  remain invariant and the other case is that they transform to the action  $S$  and the Hamiltonian  $H$ .

For the case that  $S'$  and  $H'$  remain invariant, the matrix  $\mathcal{M}_n$  is orthogonal, i.e.,

$$\mathcal{M}_n^T \mathcal{M}_n = \mathcal{M}_n \mathcal{M}_n^T = \mathbf{1}. \quad (34)$$

When  $S'$  and  $H'$  transform to the action  $S$  and the Hamiltonian  $H$ , the matrix  $\mathcal{M}_n$  satisfies the equations

$$\mathcal{M}_n^T \mathcal{M}_n = \mathcal{M}_n \mathcal{M}_{-n}^T = \mathbf{J}. \quad (35)$$

Note that the matrices  $\mathcal{M}_n$  and  $\mathbf{J}$  satisfy the identity

$$\mathcal{M}_n \mathbf{J} = \mathbf{J} \mathcal{M}_{-n}. \quad (36)$$

Therefore, we can write the equations (28) and (35) in various forms. This identity also implies that  $\det(\mathcal{M}_n) = \det(\mathcal{M}_{-n})$ .

One may consider the action  $\bar{S} = S + S'$ . The equation of motion extracted from the action  $\bar{S}$  is the equation (17). In the next sections we shall consider only the action  $S$ .

## 4 Closed string

The solution of the equation of motion (17) for closed string is

$$X_n^I(\tau, \sigma) = x_n^I \cos(\mu_n \tau) + \frac{p_n^I}{p^+} \frac{\sin(\mu_n \tau)}{\mu_n} + \frac{i}{2} \sqrt{2\alpha'} \sum_{l \neq 0} \frac{1}{W_{ln}} \left( \alpha_{ln}^I e^{-i(\omega_{ln} \tau - \bar{l} \sigma)} + \tilde{\alpha}_{ln}^I e^{-i(\omega_{ln} \tau + \bar{l} \sigma)} \right), \quad (37)$$

where  $W_{ln}$ ,  $\omega_{ln}$  and  $\bar{l}$  are given by

$$W_{ln} = \alpha' p^+ \omega_{ln} = \text{sgn}(l) \sqrt{l^2 + (\alpha' m_n)^2} \quad , \quad \bar{l} = \frac{l}{\alpha' p^+}. \quad (38)$$

In addition, we have  $m_n = p^+ \mu_n$ .

The condition (6) implies that

$$\begin{aligned} x_n^{I\dagger} &= x_{-n}^I, \\ p_n^{I\dagger} &= p_{-n}^I, \\ \alpha_{ln}^{I\dagger} &= \alpha_{-l, -n}^I, \\ \tilde{\alpha}_{ln}^{I\dagger} &= \tilde{\alpha}_{-l, -n}^I, \end{aligned} \quad (39)$$

where  $l$  is non-zero integer and  $n$  is any integer number. In other words, conjugation changes the signs of both indices  $l$  and  $n$ . Note that  $\alpha_{-l,-n}^I$  and  $\tilde{\alpha}_{-l,-n}^I$  for positive  $l$  and  $n$  are creation operators, while  $\alpha_{ln}^I$  and  $\tilde{\alpha}_{ln}^I$  are annihilation operators. Furthermore,  $\alpha_{-l,n}^I$  and  $\tilde{\alpha}_{-l,n}^I$  with respect to the index  $l$  are creation operators, while with respect to the index  $n$  are annihilation operators. Similar interpretation exists for the operators  $\alpha_{l,-n}^I$  and  $\tilde{\alpha}_{l,-n}^I$ .

The quantization (21) leads to the algebras

$$\begin{aligned} [x_n^I, p_{n'}^J] &= i\delta^{IJ}\delta_{n+n',0}, \\ [\alpha_{ln}^I, \alpha_{l'n'}^J] &= [\tilde{\alpha}_{ln}^I, \tilde{\alpha}_{l'n'}^J] = W_{ln}\delta^{IJ}\delta_{l+l',0}\delta_{n+n',0}. \end{aligned} \quad (40)$$

For simplification consider

$$\begin{aligned} \alpha_{ln}^I &= \sqrt{|W_{ln}|}a_{ln}^I, \\ \tilde{\alpha}_{ln}^I &= \sqrt{|W_{ln}|}\tilde{a}_{ln}^I. \end{aligned} \quad (41)$$

The operators  $a_{ln}^I$  and  $\tilde{a}_{ln}^I$  satisfy

$$\begin{aligned} a_{ln}^{I\dagger} &= a_{-l,-n}^I, \\ \tilde{a}_{ln}^{I\dagger} &= \tilde{a}_{-l,-n}^I. \end{aligned} \quad (42)$$

These operators have the algebra

$$[a_{ln}^I, a_{l'n'}^J] = [\tilde{a}_{ln}^I, \tilde{a}_{l'n'}^J] = \text{sgn}(l)\delta^{IJ}\delta_{l+l',0}\delta_{n+n',0}. \quad (43)$$

In terms of these oscillators, the Hamiltonian (20) for closed strings can be written as  $H = H_0 + H_{\text{osc}}$ , where

$$H_{\text{osc}} = \sum_{n \in \mathbb{Z}} \sum_{l=1}^{\infty} \left( |\omega_{ln}| (a_{-l,-n}^I a_{ln}^I + \tilde{a}_{-l,-n}^I \tilde{a}_{ln}^I) \right) + A + \tilde{A}, \quad (44)$$

for the oscillating part. With respect to the index  $l$  this part of the Hamiltonian is normal ordered. The constants  $A$  and  $\tilde{A}$  come from the algebra (43), and they are

$$A = \tilde{A} = 4 \sum_{n \in \mathbb{Z}} \sum_{l=1}^{\infty} |\omega_{ln}| = -\frac{1}{3\alpha'p^+} + 8 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} |\omega_{ln}|, \quad (45)$$

where we used  $\sum_{l=1}^{\infty} l \rightarrow -\frac{1}{12}$ . According to the equation (42), for each string (i.e., for each  $n$ )  $H_{\text{osc}}^{(n)}$  is Hermitian.

The zero mode part of the closed strings Hamiltonian has the form

$$H_0 = \frac{1}{2p^+} \sum_{n \in \mathbb{Z}} (p_n^I p_{-n}^I + m_n^2 x_n^I x_{-n}^I). \quad (46)$$

The equations in (39) imply that for each string,  $H_0^{(n)}$  also is Hermitian. Let us define the operators  $a_{0n}^I$  and  $a_{0n}^{I\dagger}$  as

$$\begin{aligned} a_{0n}^I &= \frac{1}{\sqrt{2|m_n|}}(p_n^I - i|m_n|x_n^I), \\ a_{0n}^{I\dagger} &= \frac{1}{\sqrt{2|m_n|}}(p_{-n}^I + i|m_n|x_{-n}^I). \end{aligned} \quad (47)$$

Therefore, we obtain

$$H_0 = \sum_{n \in \mathbb{Z}} (\omega_{0n} a_{0n}^{I\dagger} a_{0n}^I) + A_0. \quad (48)$$

This form of  $H_0$  is similar to  $H_{\text{osc}}$ . The equation (38) gives  $\omega_{0n} = |\mu_n|$ . The constant  $A_0$  only depends on the radius of compactification

$$A_0 = 4 \sum_{n \in \mathbb{Z}} \omega_{0n} = -\frac{2}{3R} = -\frac{8}{3}\pi^2 \alpha' T_2, \quad (49)$$

where we used the expression (14) for the radius  $R$ . The operators in the zero mode part satisfy the algebra

$$[a_{0n}^I, a_{0n'}^{J\dagger}] = \delta^{IJ} \delta_{nn'}. \quad (50)$$

In fact, for  $n = n'$  this is the harmonic oscillator algebra.

## 5 Open string

Variation of the action (15) gives the following boundary conditions for the open string with the mass number  $n$ ,

$$\begin{aligned} (\partial_\sigma X_n^i)_{\sigma_0} &= 0, \quad \text{for the Neumann directions,} \\ (\partial_\tau X_n^a)_{\sigma_0} &= 0, \quad \text{for the Dirichlet directions,} \end{aligned} \quad (51)$$

where  $\sigma_0 = 0, 2\pi\alpha'p^+$  shows the ends of the open string. According to these conditions, the solutions of the equation of motion (17) are

$$X_n^i(\tau, \sigma) = x_n^i \cos(\mu_n \tau) + \frac{p_n^i}{p^+} \frac{\sin(\mu_n \tau)}{\mu_n} + i\sqrt{2\alpha'} \sum_{l \neq 0} \left( \frac{1}{W_{ln}} \alpha_{ln}^i e^{-i\omega_{ln}\tau} \cos\left(\frac{l\sigma}{2\alpha'p^+}\right) \right), \quad (52)$$

$$X_n^a(\tau, \sigma) = \sqrt{2\alpha'} \sum_{l \neq 0} \left( \frac{1}{W_{ln}} \alpha_{ln}^a e^{-i\omega_{ln}\tau} \sin\left(\frac{l\sigma}{2\alpha'p^+}\right) \right), \quad (53)$$

where  $W_{ln}$  and  $\omega_{ln}$  have definitions

$$W_{ln} = 2\alpha' p^+ \omega_{ln} = \text{sgn}(l) \sqrt{l^2 + (2\alpha' m_n)^2}. \quad (54)$$

Note that these are different from  $W_{ln}$  and  $\omega_{ln}$  of the closed string, that have been given by the equation (38). The modes of open string also satisfy the equations

$$\begin{aligned} x_n^{i\dagger} &= x_{-n}^i, \\ p_n^{i\dagger} &= p_{-n}^i, \\ \alpha_{ln}^{I\dagger} &= \alpha_{-l, -n}^I, \quad I \in \{i, a\}. \end{aligned} \quad (55)$$

The algebra of the open string also can be obtained from the canonical quantization (21), i.e.,

$$\begin{aligned} [x_n^i, p_{n'}^j] &= i\delta^{ij}\delta_{n+n', 0}, \\ [a_{ln}^I, a_{l'n'}^J] &= \text{sgn}(l)\delta^{IJ}\delta_{l+l', 0}\delta_{n+n', 0}, \quad I, J \in \{i, a\}, \end{aligned} \quad (56)$$

where we have  $\alpha_{ln}^I = \sqrt{|W_{ln}|} a_{ln}^I$ .

The Hamiltonian of the system is  $H = H_0 + H_{\text{osc}}$ , where

$$H_0 = \frac{1}{2p^+} \sum_{n \in Z} (p_n^i p_{-n}^i + m_n^2 x_n^i x_{-n}^i), \quad (57)$$

for the zero mode part. Using the definitions in (47),  $H_0$  can be written as

$$H_0 = -\frac{p}{12R} + \sum_{n \in Z} \omega_{0n} a_{0n}^{i\dagger} a_{0n}^i, \quad (58)$$

where  $\omega_{0n} = |\mu_n|$ . Also  $p$  shows the number of the Neumann directions. If all directions obey the Neumann boundary condition, there is  $p = 8$ . In this case,  $H_0$  for the closed and open strings is the same, as expected.

For the oscillating part we have

$$H_{\text{osc}} = \frac{1}{2} \sum_{n \in Z} \sum_{l \neq 0} \left( |\omega_{ln}| a_{-l, -n}^I a_{ln}^I \right). \quad (59)$$

Normal ordered form of this part with respect to the index  $l$  is

$$H_{\text{osc}} = C + \sum_{n \in Z} \sum_{l=1}^{\infty} \left( |\omega_{ln}| a_{-l, -n}^I a_{ln}^I \right), \quad (60)$$

where the constant  $C$  is

$$C = -\frac{1}{6\alpha' p^+} + 8 \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} |\omega_{ln}|. \quad (61)$$

The Hamiltonians (58) and (60) imply that, for each  $n$  (i.e., for each string)  $H_0^{(n)}$  and  $H_{\text{osc}}^{(n)}$ , separately are Hermitian.

## 6 Conclusions

By compactifying the bosonic membrane we obtained the action of infinite number of interacting strings (i.e., the action  $S$ ) in the quantized pp-wave backgrounds. The radius of the compactification is fixed and has an expression in terms of the parameters of the string and membrane theories.

We found some transformations that the above action remains invariant, or transforms to another action, which was called “mass dual action”. The corresponding Hamiltonian (i.e.,  $H$ ) also transformed as like as the action  $S$ . Under some other transformations the mass dual action and its corresponding Hamiltonian remain invariant or change to the action  $S$  and the Hamiltonian  $H$ .

We obtained quantization of the massive strings from the membrane quantization. This implies that the quantized pp-wave background is proportional to the membrane tension and the slope  $\alpha'$ . By the mass-duality, the quantization of the mass dual action  $S'$  was obtained from the quantization of the initial action  $S$ .

The modes of both closed and open strings depend on the mass number “ $n$ ”. Under the conjugation of the massive string coordinates, the mass index of the modes changes its sign, i.e.,  $n \rightarrow -n$ . This implies that a string oscillator with respect to this index can be creation or annihilation operator. Therefore, in the algebras, strings modes with opposite but equal absolute values of the mass indices, have conjugation with each other.

The Hamiltonians of both closed and open strings in terms of the strings modes have been obtained. These Hamiltonians with respect to one of the indices of the oscillators are normal ordered. As expected, for each string the zero mode part and the oscillating part of the Hamiltonian are Hermitian.

## References

- [1] D. Amati and C. Klimcik, Phys. Lett. **B210**(1988)92; G.T. Horowitz and A.R. Steif, Phys. Rev. Lett. **64**(1990)260; H.J. de Vega and N. Sanchez, Phys. Rev. Lett. **65**(1990)1517, Phys. Lett. **B244**(1990)215; O. Jofre and C. Nunez, Phys. Rev. **D50**(1994)5232, hep-th/9311187.
- [2] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, JHEP **0201**(2002)047, hep-th/0110242.

- [3] R.R. Metsaev, Nucl. Phys. **B625**(2002)70, hep-th/0112044; R.R. Metsaev and A.A. Tseytlin, Phys. Rev. **D65**(2002)126004, hep-th/0202109.
- [4] C.R. Nappi and E. Witten, Phys. Rev. Lett. **71**(1993)3751, hep-th/9310112; E. Kiritsis and C. Kounnas, Phys. Lett. **B320**(1994)264, hep-th/9310202; C. Klimcik and A.A. Tseytlin, Phys. Lett. **B323**(1994)305, hep-th/9311012; K. Sfetsos and A.A. Tseytlin, Nucl. Phys. **B427**(1994)245, hep-th/9404063; J.G. Russo and A.A. Tseytlin, Nucl. Phys. **B448**(1995)293, hep-th/9411099, JHEP **0204**(2002)021, hep-th/0202179.
- [5] G.T. Horowitz and A.R. Steif, Phys. Rev. **D42**(1990)1950; G.T. Horowitz and A.A. Tseytlin, Phys. Rev. **D51**(1995)2896, hep-th/9409021; R. Brooks, Mod. Phys. Lett. **A6**(1991)841; H.J. de Vega and N. Sanchez, Phys. Rev. **D45**(1992)2783; H.J. de Vega, M. Ramon Medrano and N. Sanchez, Class. Quant. Grav. **10**(1993)2007; G. Papadopoulos, J.G. Russo and A.A. Tseytlin, hep-th/0211289; J.G. Russo and A.A. Tseytlin, JHEP **0209**(2002)035, hep-th/0208114.
- [6] R. Penrose, “*Any space-time has a plane wave as a limit*”, in Differential geometry and relativity, pp.271-275, Reidel, Dordrecht, (1976); R. Guven, Phys. Lett. **B482**(2000)255, hep-th/0005061.
- [7] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, Class. Quant. Grav. **19**(2002)L87, hep-th/0201081; M. Blau, J. Figueroa-O’Farrill and G. Papadopoulos, Class. Quant. Grav. **19**(2002)4753, hep-th/0202111; M. Hatsuda, K. Kamimura and M. Sakaguchi, Nucl. Phys. **B632**(2002)114, hep-th/0202190.
- [8] R.R. Metsaev and A.A. Tseytlin, Phys. Rev. **D63**(2001)046002, hep-th/0007036; R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, Nucl. Phys. **B596**(2001)151, hep-th/0009171; H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Phys. Rev. **D32**(1985)389.
- [9] D. Berenstein, J. Maldacena and H. Nastase, JHEP **0204**(2002)013, hep-th/0202021.
- [10] E. Bergshoeff, E. Sezgin, and P.K. Townsend, Phys. Lett. **B189** (1987)75, Ann. Phys. **185**(1988)330; M.J. Duff, hep-th/9611203.
- [11] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305**(1988)545; B. de Wit, hep-th/9902149; W. Taylor, Rev. Mod. Phys. **73**(2001)419, hep-th/0101126; A. Dasgupta, H. Nicolai and J. Plefka, Grav. Cosmol. **8** (2002)1, hep-th/0201182.
- [12] M.J. Duff, T. Inami, C.N. Pope, E. Sezgin and K.S. Stelle, Nucl. Phys. **B297**(1988)515; K.S. Stelle, “*SUPERMEMBRANES*”.

[13] M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, Phys. Lett. **B191** (1987)70.

[14] J. Polchinski, “String Theory”, *Cambridge, UK: University Press (1998)*.